1 Sheaf Spaces

Proposition 1.1. *Let X be a topological space, F a sheaf on X, $U \subset X$ open, $s, s' \in F(U)$. The following holds:*

$$s = s' \iff s_x = s'_x \ \ \forall x \in U$$

Proof. „ $\implies$ “ is clear.
„ $\impliedby$ “: By definition of direct limit, we have $s_x = s'_x \iff \exists U_x \subset U$ with $\rho^U_{U_x}(s) = \rho^U_{U_x}(s')$. Apply the monopresheaf condition on $U = \cup_{x \in U} U_x$ to see $s = s'$. □

Remark 1.2. 1.1 in general doesn't hold for presheaves. Consider for example $X = \{0, 1\}, F(X) = \{a, b\}, F(U) = \{0\}$, for $U \neq X$, with ρ constant, except ρ^X_X. For a sheaf however, this allows us to think of a sheaf as a collection of functions with values in its stalks.

Definition 1.3. Let X be a topological space. A sheaf space over X is a pair (E, p) consisting of a topological space E and a local homeomorphism $p : E \to X$. (This also forces p to be continuous.)
A morphism of sheaf spaces $f : (E, p) \to (E', p')$ is a continuous map $f : E \to E'$, s.t. $p = p' \circ f$.

Construction 1.4. Let E be a sheaf space. We will construct a sheaf of sets ΓE in a natural way, i.e. in such a way, that a morphism $f : E \to E'$ of sheaf spaces gives rise to a morphism $\Gamma f : \Gamma E \to \Gamma E'$ of sheaves.
For $U \subset X$ open, we set

$$\Gamma(U, E) := \{\sigma : U \to E \text{ cont.} | p \circ \sigma = id_U\}$$

A restriction maps, we use the usual restriction maps. Then one may show, that the map $\Gamma E : U \mapsto \Gamma(U, E)$ defines a sheaf.
Now let $f : E \to E'$ be a morphism of sheaf spaces. We have the map

$$\Gamma(U, E) \to \Gamma(U, E'), \quad \sigma \mapsto f \circ \sigma$$

and since $p = p' \circ f$, this is well-defined and gives us a morphism of sheaves $\Gamma f : \Gamma E \to \Gamma E'$.

Lemma 1.5. *Let (E, p) be a sheaf space over X. Then:*

a) p is an open map.

b) For $U \subset X$ open, $\sigma \in \Gamma(U, E)$, $\sigma(U)$ is open in E. Furthermore sets of this form give a basis for the topology of E.

c) Let (E', p') be another sheaf space, $\varphi : E \to E'$ s.t. $p = p' \circ \varphi$, p, p' local homeomorphisms. Then the following holds:

$$\varphi \text{ cont.} \iff \varphi \text{ open} \iff \varphi \text{ local homeom.}$$

Proof. a) Let $W \subset E$ be open and $x \in p(W)$. Pick an $e \in W \cap p^{-1}(x)$. Then, by the definition of a sheaf space, there exists an open neighbourhood $W' \subset W$ of e, with $p(W) \supset p(W') \ni x$ open in X.

b) Let $e \in \sigma(U)$. Then there exists an open neighbourhood $W \subset E$, s.t. $p|_W$ is a homeomorphism onto an open set $V \subset X$. Then $p|_W$ maps $W \cap \sigma(U)$ bijectively onto $U \cap V$ ($p \circ \sigma = id_U$), which is open in X. Therefore $W \cap \sigma(U)$ is an open neighbourhood of e inside $\sigma(U)$.
Let $W \subset E$ be open. Then $p(W)$ is open in X by a). Let $y \in W$. Then there exist $N \ni y$, $U \ni p(y)$ open, s.t. $p|_N : N \to U$ is a homeomorphism. Take the inverse $\sigma : U \to N \hookrightarrow E$ and restrict it to $U \cap p(W)$. Then $\sigma \in \Gamma(U \cap p(W), E)$ and $\sigma(U \cap p(W))$ is an open neighbourhood of y contained in W.

c) Local homeomorphisms are always continuous and therefore also open by a).
„φ cont. $\implies$ φ local homeom.“: Let $y \in E$. Since p' is a local homeomorphism, there exist open $N' \subset E', V \subset X$ $p'|_{N'} : N' \to V$ is a homeomorphism and $\varphi(y) \in N'$. Also, $\varphi^{-1}(N')$ is open in E. We therefore find an open $N \subset \varphi^{-1}(N')$ containing y, which p maps homeomorphically onto an open $U \subset V$. Set $N'' = p'^{-1}(U) \cap N'$ to obtain the following commuative diagram

$$\begin{array}{ccc} N & \xrightarrow{\varphi|_N} & N'' \\ \downarrow{\scriptstyle p|_N} & \swarrow{\scriptstyle p'|_{N''}} & \\ U & & \end{array}$$

with N, N'', U open and $p|_N, p'|_{N''}$ homeomorphisms. Therefore $\varphi|_N$ is also a homeomorphism.
„φ open $\implies$ φ local homeom.“: Let $y \in E$. Then there exist an open neighbourhood N of y and an open $U \subset X$, s.t. $p|_N : N \to U$ is a homeomorphism. Also $\varphi(N)$ is open, so there exist an open neighbourhood N' of $\varphi(y)$ and $V \subset U$ open, s.t. $p'|_{N'} : N' \to V$ is a homeomorphism. Set $N'' = p^{-1}(V) \cap N$ to obtain the homeomorphism $p|_{N''} : N'' \to V$ and the commuative diagram

$$\begin{array}{ccc} N'' & \xrightarrow{\varphi|_{N''}} & N' \\ \downarrow{\scriptstyle p|_{N''}} & \swarrow{\scriptstyle p'|_{N'}} & \\ V & & \end{array}$$

As before, we conclude, that $\varphi|_{N''}$ is a homeomorphism.

□

Proposition 1.6. *Let (E, p) be a sheaf space an $x \in X$. There exists a natural bijection*

$$\varphi_x : (\Gamma E)_x \to p^{-1}(x)$$

and $p^{-1}(x)$ has the discrete topology as a subspace of E.

Proof. For an open neighbourhood U of x, consider the map

$$\varphi_U : \Gamma(U, E) \to p^{-1}(x), \quad \sigma \mapsto \sigma(x).$$

These maps are compatible with restrictions and by universal property of the direct limit give rise to a map $\varphi_x : (\Gamma E)_x \to p^{-1}(x)$.
Surjectivity: Let $e \in p^{-1}(x)$. Since p is a local homeomorphism, e has an open neighbourhood $W \subset E$, s.t. $p|_W : W \to U$ is a homeomorphism for an open $U \subset X$. Consider $\sigma := (p|_W)^{-1} \in$

$\Gamma(U, E)$. Then $\varphi_x(\sigma_x) = e$.
Injectivity: Let $s \in \Gamma(U, E), t \in \Gamma(V, E)$ agree at x. By Lemma 1.5 $W := s(U) \cap t(V)$ is open in E and s, t agree on $p[W]$ since both are inverses of $p|_W$ $(s(p(s(u))) = s(u))$. Also $p(W)$ is open by Lemma 1.5, so $\rho^U_{p(W)}(s) = \rho^V_{p(W)}(t) \in \Gamma(p(W), E)$, which shows the injectivity.
For $e \in p^{-1}(x)$ and W as in the proof of surjectivity, we have $W \cap p^{-1}(x) = \{e\}$. $\square$

Remark 1.7. One easily verifies the following:

a) Γ is functorial: $\Gamma(f \circ g) = \Gamma(f) \circ \Gamma(g), \Gamma(id) = id$.

b) $f : E \to E'$ a morphism of sheaf spaces. Then the maps $(\Gamma f)_x : (\Gamma E)_x \to (\Gamma E')_x$ and $f|_{p^{-1}(x)} : p^{-1}(x) \to p'^{-1}(x)$ are isomorphic.

Construction 1.8. Let F be a presheaf on X. We will construct a sheaf space LF in a natural way, i.e. in such a way, that a morphism $f : F \to F'$ of presheaves gives rise to a morphism $Lf : LF \to LF'$ of sheaf spaces.
Set $LF := \coprod_{x \in X} F_x$ (disjoint union of the stalks of F) with $p : LF \to X$ the natural projection map $(p^{-1}(x) = F_x)$.
We give LF the following topology: For $U \subset X$ open and $s \in F(U)$, we define the map

$$\hat{s} : U \to LF, \quad x \mapsto s_x \in F_x \subset LF$$

and declare sets of the form $\hat{s}(U) = \{s_x \in LF | x \in U\}$ to be open sets. Then $\bigcup_{U \subset X \text{ open}} \{\hat{s}(U) | s \in F(U)\}$ forms a basis for the topology it generates:
Let $e \in \hat{s}(U) \cap \hat{t}(V)$ for $s \in F(U), t \in F(V)$. Then $e = s_x = t_x$ for an $x \in X$ and there therefore exists an open $W \subset U \cap V$ s.t. $\rho^U_W(s) = \rho^V_W(t)$, which means, that e has a basis neighbourhood $\widehat{\rho^U_W(s)}(W) = \widehat{\rho^V_W(t)}(W) \subset \hat{s}(U) \cap \hat{t}(V)$.
Furthermore p is continuous with respect to this topology on LF, since for $U \subset X$ open we have

$$p^{-1}(U) = \bigcup_{V \subset U \text{ open},\ s \in F(V)} \hat{s}(V).$$

Also, p is a local homeomorphism, since on $\hat{s}(U)$ it has the continuous inverse $\hat{s}$.
Now let $f : F \to F'$ be a morphism of presheaves. Then f gives rise to stalk maps $f_x : F_x \to F'_x$ and thus to a map $Lf : LF \to LF'$ s.t. $p = p' \circ Lf$. Also, LF is open, since $Lf(\hat{s}(U)) = \widehat{f(U)(s)}(U)$. So LF is continuous by Lemma 1.5.

Remark 1.9. One easily verifies, that L is functorial: $L(f \circ g) = Lf \circ Lg, L(id) = id$.

The question arises, what happens when we apply L and Γ in succession. In the first case, not much does happen:

Theorem 1.10. *Let E be a sheaf space over X. Then $(L\Gamma E, p') \cong (E, p)$.*

Proof. Let $x \in X$. By Proposition 1.6 the fibre $p^{-1}(x)$ stands in bijection to the stalk $(\Gamma E)_x$, which is $p'^{-1}(x) \subset L\Gamma E$, by Construction 1.8. By fitting these bijections together for varying $x \in X$, we obtain a bijection $\varphi : E \to L\Gamma E$ s.t. the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & L\Gamma E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & X & \end{array}$$

Let $U \subset X$ be open and $\sigma \in \Gamma(U, E)$. Then $\varphi(\sigma(U)) = \hat{\sigma}(U)$, since for $x \in U$ we have $\varphi(\sigma(x)) = \sigma_x$ (recall the construction of the bijection in Proposition 1.6). Therefore φ is open and by Lemma 1.5 also continuous. □

2 The sheafification of a presheaf

Given a presheaf F over X we can first apply L and then Γ to obtain a sheaf ΓLF, called the sheafification of F. This comes with a morphism of presheaves $n_F : F \to \Gamma LF$: For $U \subset X$ open we define

$$n_F(U) : F(U) \to \Gamma(U, LF), \quad s \mapsto \hat{s}.$$

This construction satisfies the following universal property:

Theorem 2.1 (Universal property of ΓL)**.** *Let F be a presheaf and G a sheaf over X. Then every morphism of presheaves $f : F \to G$ factors uniquely through ΓLF, i.e. there exists a unique sheaf morphism $g : \Gamma LF \to G$, s.t. the following diagram commutes:*

$$\begin{array}{ccc} F & \xrightarrow{\quad f \quad} & G \\ & {\scriptstyle n_F}\searrow \quad \nearrow{\scriptstyle g} & \\ & \Gamma LF & \end{array}$$

This theorem shows that ΓLF is „the best “ sheaf we can make out of F. For the proof we will need two further lemmata:

Lemma 2.2. *Let G be a presheaf. Then G is a sheaf $\iff$ $n_G : G \to \Gamma LG$ is an isomorphism of presheaves.*

Proof. „ $\Longleftarrow$ “ is clear since ΓLG is a sheaf.
„ $\Longrightarrow$ “: We check, that each $G(U) \to \Gamma(U, LG)$, $s \mapsto \hat{s}$ is bijective.
Injectivity is clear by Proposition 1.1.
Surjectivity: For $t \in \Gamma(U, LG)$, $t(U)$ is open in LG by Lemma 1.5. Therefore, for each $x \in U$, $t(x) \in G_x$ has a basic neigbourhood inside $t(U)$ of the form $\widehat{s^x}(U_x)$ for an open $U_x \subset U$ and $s \in G(U_x)$. (Recall Construction 1.8.) The s^x satisfy the glueing condition, since for $x, y \in U, V := U_x \cap U_y$, $\rho_V^{U_x}(s^x)$ and $\rho_V^{U_y}(s^y)$ have the same germ everywhere ($(\rho_V^{U_x}(s^x))_z = s_z^x = t(z)$ for $z \in V$). Therefore, since G is a sheaf, there exists an $s \in G(U)$ s.t. $s_x = (s^x)_x = t(x)$ for all $x \in U$, so $\hat{s} = t$. □

Remark 2.3. $n_F : F \to \Gamma LF$ is natural in the sense, that if $f : F \to F'$ is a morphism of presheaves, the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{n_F} & \Gamma LF \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Gamma Lf} \\ F' & \xrightarrow{n_{F'}} & \Gamma LF' \end{array}$$

Lemma 2.4. *Let F be a presheaf on X. The maps $n_{F,x} : F_x \to (\Gamma LF)_x$ induced on the stalks by n_F are isomorphisms.*

Proof. We have the following diagramm:

$$\begin{array}{ccc} F(U) & \xrightarrow{n_F} & \Gamma(U, LF) \\ \downarrow{\scriptstyle (\;)_x} & \swarrow{\scriptstyle \varphi_U} & \downarrow{\scriptstyle (\;)_x} \\ F_x & \underset{\varphi_x}{\overset{n_{F,x}}{\rightleftarrows}} & (\Gamma LF)_x \end{array}$$

φ_x and φ_U are the maps from Proposition 1.6. (Recall that $F_x = p^{-1}(x)$ by Construction 1.8.) The following identities hold: $n_{F,x} \circ (\;)_x = (\;)_x \circ n_F, \varphi_U \circ n_F = (\;)_x, \varphi_x \circ (\;)_x = \varphi_U$.
It suffices to show $n_{F,x} = \varphi_x^{-1}$, since by Proposition 1.6 φ_x is an isomorphism. By universal property of the direct limit, this is equivalent to showing $(\;)_x \circ n_F = \varphi_x^{-1} \circ (\;)_x$, which again is equivalent to showing $\varphi_x^{-1} \circ \varphi_U \circ n_F = \varphi_x^{-1} \circ (\;)_x$ since $(\;)_x = \varphi_x^{-1} \circ \varphi_U$. This however is true, because $\varphi_U \circ n_F = (\;)_x$. □

We are now ready to prove Theorem 2.1:

Proof. [of Theorem 2.1] Suppose there is a sheaf morphism $g : \Gamma Lf \to G$ s.t. $f = g \circ n_F$. Then $f_x = (g \circ n_F)_x = g_x \circ n_{F,x}$, so $g_x = n_{F,x}^{-1} \circ f_x$ since $n_{F,x}$ is an isomorphism by Lemma 2.4. This shows the uniqueness of g.
Existence: By Remark 2.3 we have the following commutative diagram:

$$\begin{array}{ccc} F & \xrightarrow{n_F} & \Gamma LF \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Gamma Lf} \\ G & \xrightarrow{n_G} & \Gamma LG. \end{array}$$

Since G is a sheaf, n_G is an isomorphism by Lemma 2.2. We can therefore set $g := n_G^{-1} \circ \Gamma Lf$ and are finished. □

Example 2.5 (The constant sheaf)**.** Let A be a set and X a topological space. Recall that the constant presheaf A_X on X was given by $A_X(U) = A$ for $U \subseteq X$ open and $\rho_V^U = id_A : A_X(U) \to A_X(V)$ for an open subset $V \subseteq U$.
We first apply L and obtain the sheaf space $LA_X \xrightarrow{p} X$ s.t. $p^{-1}(x) = A_{X_x} = A$ for all $x \in X$. As sets we therefore have $LA_X = A \times X$ and $p = \pi_2$.
For $U \subset X$ open $a \in A_X(U) = A$, we have:

$$\hat{a}(U) = \{a_x \in A \times X |\; x \in U\} = \{a\} \times U$$

So by Construction 1.8 the topology on $A \times X$ has as a basis sets of the form $\{a\} \times U$ for $a \in A$ and $U \subset X$ open.
The topology on $A \times X$ therefore is the product-topology with A given the discrete topology. We now consider $F := \Gamma LA_X$. The sections are given by

$$F(U) = \Gamma(U, LA_X) = \{\sigma : U \to A \times X \; cont. \mid \pi_2 \circ \sigma = id_U\}$$
$$\cong \{s : U \to A \; cont.\}$$
$$= \{s : U \to A \; locally\ constant\}$$

The isomorphism ist trivial since $\pi_2 \circ \sigma = id_U$ means σ is uniquely determined by $\pi_1 \circ \sigma$.
The equality of the latter two sets can be concluded as follows:
Let $x \in U$ and $s : U \to A$ be continuous. Then $s^{-1}(s(x))$ is open, contains x and s is obviously constant on $s^{-1}(s(x))$.
If conversely $s : U \to A$ is locally constant and $a \in A$. Then for each $x \in s^{-1}(a)$ we can find an open neighourbood $U_X \subseteq U$ of x with $s(U_X) = a$.

Now if U is disconnected and A has > 1 element, we have that n_{A_X} is no isomorphism, so A_X was not originally a sheaf by Lemma 2.2.

Definition 2.6. The constant sheaf over X modelled on A is the sheaf whose sheaf space is $A \times X \xrightarrow{\pi_2} A$ (A given the discrete topology) and is also denoted by A_X.

Remark 2.7. We can use the concept of sheaf spaces to better understand sheaves of abelian groups by studying their sheaf space. In particular, if F is a sheaf of abelian groups, then the corresponding sheaf space (LF, p) has the property, that each fibre $p^{-1}(x)$ has the structure of an abelian group. And since p is continuous, these groups vary continuously in some sense, for varying x.